ON FUNDAMENTAL GROUPS OF SYMPLECTICALLY ASPHERICAL MANIFOLDS II: ABELIAN GROUPS

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ABSTRACT. We describe all abelian groups which can appear as the fundamental groups of closed symplectically aspherical manifolds. The proofs use the theory of symplectic Lefschetz fibrations.

1. INTRODUCTION

The present paper is a continuation of [IKRT]. Recall that a symplectic form ω on a smooth manifold M is called *symplectically aspherical* if

$$\int_{S^2} f^*\omega = 0$$

for any continuous map $f : S^2 \to M$. This is expressed equivalently by saying that the cohomology class $[\omega] \in H^2(M; \mathbb{R})$ vanishes on the image of the Hurewicz map $h : \pi_2(M) \to H_2(M; \mathbb{R})$. A *symplectically aspherical manifold* is defined as a connected manifold that admits a symplectically aspherical form.

Gompf asked a question about the topology of such manifolds in [G2]. The importance of this class of symplectic manifolds comes from the Floer theory which is much simpler in the symplectically aspherical case [F, S]. In [IKRT] and in this article we are interested in the following question (cf. [G2]).

Question 1.1. What groups can be realized as fundamental groups of closed symplectically aspherical manifolds?

In the sequel we call these groups *symplectically aspherical*. One of our main results is the classification of finitely generated abelian symplectically aspherical groups.

Theorem 1.2. *A finitely generated abelian group Γ is symplectically aspherical if and only if either $\Gamma \cong \mathbb{Z}^2$ or* $\mathrm{rank}(\Gamma) \geq 4$.

Realization Question 1.1 can be thought of in the general context of possible restrictions which an additional geometric structure on a given manifold imposes on its algebraic topology and, in particular, on its fundamental group. For example, we could mention the Hopf question on fundamental groups of 3-manifolds or Serre's question about fundamental groups of projective varieties (see Problem on page 10 in [Se] or [ABCKT, Open Problem 1.17]). The same question for compact Kähler manifolds is still unsolved (see [ABCKT] for an exposition).

In [G2] Gompf showed that any finitely presentable group can be realized as the fundamental group of a closed symplectic manifold. On the other hand, this general realization result fails under the condition of symplectic asphericity: for instance, finite groups and $\mathbb{Z}$ are not symplectically aspherical, cf. [IKRT].

Example 1.3. Let Γ be a (finitely generated) group of real cohomological dimension 3, i.e. $H^3(\Gamma; \mathbb{R}) \neq 0$ while $H^i(\Gamma; \mathbb{R}) = 0$ for $i > 3$. Then Γ is not symplectically aspherical. Indeed, if (M, ω) is a closed symplectically aspherical manifold and $\pi_1(M) = \Gamma$ then the class $[\omega]$ lies in the image of the homomorphism $c^* : H^2(K(\Gamma, 1); \mathbb{R}) \to H^2(M; \mathbb{R})$, induced by the classifying map $c : M \to K(\Gamma, 1)$. In other words, $[\omega] = c^*(a)$ for some $a \in H^2(\Gamma; \mathbb{R})$. Since Γ is three dimensional, we conclude that $[\omega]^2 = c^*(a^2) = 0$, and hence $\dim M = 2$. Therefore M is an aspherical closed oriented surface, and thus $H^3(\Gamma; \mathbb{R}) = H^3(M; \mathbb{R}) = 0$.

We see that, in particular, no finitely generated abelian group of rank 3 are symplectically aspherical. Hence our Theorem 1.2 states that the obvious necessary condition is also sufficient for a finitely generated abelian group to be symplectically aspherical.

Moreover, we answer the question from [IKRT], motivated by Gompf [G2], about the relation between two classes of symplectically aspherical groups. Recall that, the class $\mathcal{A}$ consists of groups Γ realizable as $\pi_1(M)$, where M s symplectically aspherical with $\pi_2(M) = 0$, while the class $\mathcal{B}$ consists of symplectically aspherical groups realizable as $\pi_1(M)$ with $\pi_2(M) \neq 0$. It is easy to see that the group $\mathbb{Z}^2$ belongs to $\mathcal{A}$ and does not belong to $\mathcal{B}$, while in [IKRT] we asked whether $\mathcal{B} \subset \mathcal{A}$. In this paper we show that $\mathcal{B} \not\subset \mathcal{A}$, namely, $\mathbb{Z}^4 \oplus \mathbb{Z}/2 \in \mathcal{B} \setminus \mathcal{A}$ (see Proposition 5.3). This phenomenon deserves further investigation, since this gives a non-realizability result, which may reveal some new essentially symplectic (non-topological) properties.

In this work we focus on 4-dimensional symplectic Lefschetz fibrations and provide conditions implying the symplectic asphericity of the

total space. The use of this technique proved to be very effective in many other contexts. In fact, any closed 4-dimensional symplectic manifold admits a Lefschetz pencil, and, therefore, becomes a symplectic Lefschetz fibration after blow-up at several points [D1, GS]. Using the Donaldson hyperplane section theorem [D2], one can reduce the realizability problem to the 4-dimensional case (see [IKRT, Proposition 2.2] for details). Explicit construction of Lefschetz fibrations with given fundamental groups was given by Amoros, Bogomolov, Katzarkov and Pantev [ABKP]. Our results can be regarded as solutions of a similar construction problem under additional restriction of symplectic asphericity.

Throughout the paper Σ_g denotes the closed orientable surface of genus g and π_g denotes the fundamental group of Σ_g.

2. Symplectic Lefschetz fibrations

Definition 2.1 ([ABKP, D1, GS])**.** Let X be a compact, connected, oriented, smooth 4-manifold, possibly with boundary. A *Lefschetz fibration structure* on X is a surjective map $f : X \to \Sigma$ where Σ is a compact, connected, oriented surface and $f^{-1}(\partial\Sigma) = \partial X$. Furthermore, the following is required:

- the set $\{q_1, \ldots, q_n\}$ of critical points of f is finite;
- $f(q_i) \neq f(q_j)$ for $i \neq j$;
- if $b \in \Sigma_g$ is a regular value of f then $f^{-1}(b)$ is a closed connected orientable surface;
- let $F_i = f^{-1}(f(q_i))$. Then there exists a complex chart $\varphi : U_i \to \mathbb{C}^2$ with $q_i \in U_i \subset X$ and a complex chart $\psi_i : V_i \to \mathbb{C}$ with $f(q_i) \in V_i \subset f(U) \subset \Sigma$ such that $\psi_i f \varphi_i^{-1} : \mathbb{C}^2 \to \mathbb{C}$ has the form $(x, y) \mapsto x^2 + y^2$. Here we require that both complex charts preserve the orientations.

Clearly, the manifolds $f^{-1}(b_1)$ and $f^{-1}(b_2)$ are diffeomorphic for any two regular values b_1, b_2 of f. We define the *regular fiber* of f to be the (diffeomorphism equivalence class of the) manifold $f^{-1}(b)$ where b is a regular value of f. Non-regular fibers are called *singular fibers*.

When we write "a Lefschetz fibration $F \to X \to B$", it means that F is the regular fiber.

Constructions 2.2. (a) Let D be a closed disc $D \subset \mathbb{C}$ about the origin and $F \to X \xrightarrow{f} D$ be a Lefschetz fibration over D with one singular point over $0 \in D$. The restriction of f onto ∂D is the locally trivial bundle that gives us and is characterized by a self-diffeomorphism $T : F \to F$ that is either the identity map or (isotopic to) the Dehn twist along a certain simple closed curve C on F. In other words, the monodromy about the origin of the disc is the (isotopy class of the) Dehn twist $T : F \to F$ along C. This curve C is a *vanishing cycle* corresponding to T, and the singular fiber of f is homeomorphic to F/C.

(b) In order to construct a Lefschetz fibration $F \to X \xrightarrow{f} S^2$ over the sphere one proceeds as follows. Following item (a), take n Lefschetz fibrations $X_1, \ldots, X_n$ over discs $D_1, \ldots, D_n$ with monodromies $T_1, \ldots, T_n$, respectively. Take the boundary connected sum D' of these discs and consider the corresponding fiber sum of the Lefschetz fibrations X_i. In this way we obtain a Lefschetz fibration $X' \to D'$ over the disc D' with monodromy over $\partial D'$ equal to the product $T_1 \cdots T_n$. Assuming that this product is isotopic to the identity and choosing an isotopy, one extends the above fibration $X' \to D'$ to a fibration $X_0 \to D_0$ over a larger disc $D_0 \supset D'$ so that the new one has trivial monodromy over the boundary. Finally, this fibration over D_0 can be extended trivially to a Lefschetz fibration over the sphere.

Each Dehn twist T_i in 2.2(b) can be chosen so that it is supported in a small tubular neighborhood of its vanishing cycle. Let $p \in F \setminus (\operatorname{supp} T_1 \cup \cdots \cup \operatorname{supp} T_n)$ be a basepoint.

Lemma 2.3. *Suppose that the product $T_1 \cdots T_n$ represents the neutral element in the mapping class group $\pi_0(\operatorname{Diff}(F, p))$ (in other words, the isotopy from $T_1 \cdots T_n$ to the identity map preserves $p \in F$). Then the Lefschetz fibration constructed in 2.2(b) admits a section. Furthermore, $\pi_1(X)$ is isomorphic to the quotient group $\pi_1(F)/N$ where N is the normal subgroup of $\pi_1(F)$ generated by vanishing cycles $C_1, \ldots, C_n$ corresponding to Dehn twists $T_1, \ldots, T_n$, respectively.*

Proof. First observe that each fibration $F \to X_i \to D_i$ over the small disc about a critical value contains the trivial subbundle $(F \setminus \operatorname{supp} T_i) \times D_i$. Let $s_i : D_i \to (F \setminus \operatorname{supp} T_i) \times D_i \subset X_i$ be the section given by $s_i(x) := (p, x)$. Together these sections yield a section $s' : D' \to (F \setminus (\operatorname{supp} T_1 \cup \cdots \cup \operatorname{supp} T_n)) \times D' \subset X'$, $s'(x) := (p, x)$. Since the isotopy from $T_1 \cdots T_n$ to the identity map preserves the basepoint p, we conclude that the section s' extends to a section $S^2 \to X$.

The last claim is proved in [ABKP, Lemma 2.3].   □

A *symplectic Lefschetz fibration* is defined to be a Lefschetz fibration with a symplectic form ω on the total space whose restriction on each of the fibers is non-degenerate, see [ABKP, GS] for details.

A proof of the following Theorem 2.4 (without item (iv)) is contained in [ABKP, Theorem A]. A weak version of this theorem can also be deduced by combining the theorem of Donaldson [D1, ADK] that every symplectic 4-manifold admits a structure of a Lefschetz fibration, and the theorem of Gompf [G1] on realizability of any finitely presentable group as the fundamental group of closed symplectic 4-manifold.

Recall that a group G is called *finitely presentable* if there exists an epimorphism $f : F \to G$ where F is a free group of finitely many free generators and $\mathrm{Ker}\, f$ is a normal closure of a finitely generated subgroup of F. Following [ABKP], we define a finite presentation of a group G to be an epimorphism $a : A \to G$ where A is a finitely presentable group and $\mathrm{Ker}\, a$ is a normal closure of a finitely generated subgroup of A.

Theorem 2.4. *Let $p_\Gamma : \pi_g \to \Gamma$ be a finite presentation of a group Γ. Then there exists an epimorphism $p_{h,g} : \pi_h \to \pi_g$ for some $h \geq g$ and a symplectic Lefschetz fibration*

$$\Sigma_h \xrightarrow{\ i\ } X \xrightarrow{\ f\ } S^2$$

with the following properties:

   (i) *$\pi_1(X) = \Gamma$;*
  (ii) *the homomorphism $i_* : \pi_h \to \pi_1(X)$ coincides with $p_\Gamma p_{h,g}$;*
 (iii) *the homomorphism $p_{h,g}$ is induced by a map $\Sigma_h \to \Sigma_g$ of non-zero degree;*
  (iv) *the map (fibration) f has a section.*

Proof. The first three items are proved in [ABKP, Theorem A]. Roughly speaking, the idea of the proof looks as follows. The authors construct a map $p_{h,g} : \pi_h \to \pi_g$ such that the map $p_\Gamma p_{h,g} : \pi_h \to \Gamma$ is a finite presentation of Γ. Moreover, the kernel of $p_\Gamma p_{h,g}$ is generated as a normal subgroup by homotopy classes of simple closed curves $C_1, \ldots, C_n$. Furthermore, the Dehn twists T_i along C_i, $i = 1, \ldots, n$ satisfy the conditions of Lemma 2.3. Now, we construct a Lefschetz fibration as in 2.2(b). This Lefschetz fibration turns out to be symplectic in view of [GS, Theorem 10.2.18] or [ABKP, Proposition 2.3].

Because of what we said above and Lemma 2.3, we get the proof of (i), (ii) and (iv).

Finally, the map $p_{h,g}$ is induced by a composition $\Sigma_h \to \Sigma_e \to \Sigma_g$, where Σ_e is obtained from Σ_g by adding handles (and the map $\Sigma_e \to \Sigma_g$ collapses these handles) and the map $\Sigma_h \to \Sigma_e$ is a finite ramified covering. This implies (iii). □

Recall that, given a map $f : X \to Y$ of spaces, a cohomology class $a \in H^k(X)$ is said to be *totally non-cohomologous to zero*, abbreviated as TNCZ, if $i_y^*(a) \neq 0$ for each inclusion $i_y : f^{-1}(y) \subset X$ where y runs over all points of Y.

Theorem 2.5. *Let ω_{Σ_g} be a symplectic form on Σ_g and*

$$F \xrightarrow{\ i\ } X \xrightarrow{\ f\ } \Sigma_g$$

be a symplectic Lefschetz fibration. Let $\Omega \in H^2(X; \mathbb{R})$ be a TNCZ class. Then there exists $C \in \mathbb{R}$ such that the cohomology class $\Omega + Cf^*[\omega_{\Sigma_g}]$ contains a symplectic form on X.

Proof. This is actually proved in [GS, Theorem 10.2.18]. Namely, without loss of generality we can assume that $\langle \Omega, [F] \rangle > 0$. Then $\langle \zeta, [F] \rangle > 0$ for each closed form $\zeta \in \Omega$. Let η be the form described in the proof of Theorem 10.2.18 in [GS]. Then $\eta \in \Omega$, while $\eta + Cf^*\omega_{\Sigma_g}$ is a symplectic form on X, see [GS, Proposition 10.2.20]. □

3. SYMPLECTICALLY ASPHERICAL LEFSCHETZ FIBRATIONS

Given a space X, we call a non-zero cohomology class $a \in H^k(X; G)$ *aspherical*, if $\langle a, f_*[S^k] \rangle = 0$ for any continuous map $f : S^k \to X$.

Proposition 3.1. *Let $p_\Gamma : \pi_g \to \Gamma$ be a finite presentation of a group Γ such that*

$$p_\Gamma^* : H^2(\Gamma; \mathbb{R}) \to H^2(\pi_g; \mathbb{R})$$

is a non-zero homomorphism. Then there exists a symplectic Lefschetz fibration $\Sigma_h \xrightarrow{i} X \xrightarrow{f} S^2$ with $\pi_1(X) = \Gamma$ and TNCZ aspherical class $\Omega \in H^2(X)$. Furthermore, this Lefschetz fibration admits a section.

Proof. Because of Theorem 2.4, there exists an epimorphism

$$p_{h,g} : \pi_h \to \pi_g$$

and a symplectic Lefschetz fibration $\Sigma_h \xrightarrow{i} X \xrightarrow{f} S^2$ admitting a section and such that $\pi_1(X) = \Gamma$ and $i_* = p_\Gamma \circ p_{h,g}$. Moreover, since $p_{h,g}$ is induced by a map $\Sigma_h \to \Sigma_g$ of non-zero degree, we conclude that

(3.1) $$p_{h,g}^* : H^2(\pi_g; \mathbb{R}) \to H^2(\pi_h; \mathbb{R})$$

is an isomorphism.

Let $c : X \to K(\Gamma, 1)$ be a map that induces an isomorphism of fundamental groups. Let $i : \Sigma_h \to X$ be the inclusion of a regular fiber. We have the commutative diagram

$$
\begin{array}{ccccccc}
\pi_h & = & \pi_1(\Sigma_h) & \xrightarrow{\ i_* \ } & \pi_1(X) & \xrightarrow[\cong]{\ c_* \ } & \pi_1(K(\Gamma, 1)) \\[2mm]
\| & & \downarrow{\scriptstyle p_{h,g}} & & \| & & \| \\[2mm]
\pi_h & \xrightarrow{\ p_{h,g} \ } & \pi_g & \xrightarrow{\ p_\Gamma \ } & \Gamma & = & \Gamma
\end{array}
$$

Since the homomorphism (3.1) is an isomorphism, we conclude that the homomorphism $i^* \circ c^* : H^2(\Gamma; \mathbb{R}) \to H^2(\pi_1(\Sigma_h); \mathbb{R})$ is nontrivial because so is $p_\Gamma^* : H^2(\Gamma; \mathbb{R}) \to H^2(\pi_g; \mathbb{R})$. Take any $a \in H^2(\Gamma; \mathbb{R})$ with $c^*(a) \neq 0$ and put $\Omega = c^*(a)$. Then $\langle \Omega, i_*[\Sigma_h] \rangle \neq 0$ because $i^*(c^*(a)) \neq 0$.

Finally, if $j : \Sigma \to X$ is the inclusion of a singular fiber then $\langle \Omega, j_*(\Sigma) \rangle \neq 0$ because singular fibers are homologous in X to regular fibers. $\qquad\square$

Lemma 3.2. *Let $F \xrightarrow{i} X \xrightarrow{f} \Sigma_g$ be a symplectic Lefschetz fibration over the base of genus $g > 0$. Suppose that X admits a TNCZ aspherical class $\Omega \in H^2(X; \mathbb{R})$. Then X is symplectically aspherical.*

Proof. Because of Theorem 2.5, there exists a symplectic structure ω on X whose cohomology class is of the form $\Omega + C f^*[\omega_{\Sigma_g}]$, for some constant $C \in \mathbb{R}$. Clearly, it is an aspherical class. $\qquad\square$

Proposition 3.3. *Let $F \to X \to \Sigma_g$ be a symplectic Lefschetz fibration that admits a TNCZ aspherical class $\Omega \in H^2(X; \mathbb{R})$. Let $Y = F \times \Sigma_h$ with the product symplectic structure. If $g + h > 0$ then the Gompf symplectic fiber sum $X \#_F Y$ is symplectically aspherical.*

Proof. It is clear that $F \to X \#_F Y \to \Sigma_g \# \Sigma_h = \Sigma_{g+h}$ is a symplectic Lefschetz fibration. Furthermore, the retraction $\Sigma_h \to D^2$ yields the retraction $\Sigma_h \times F \to D^2 \times F$. Now, the degree one map $\Sigma_g \# \Sigma_h \to \Sigma_g$ (collapsing the summand $\Sigma_h \setminus D^2$ to the point) yields the following commutative diagram:

$$
\begin{array}{ccc}
X \#_F Y & \xrightarrow{\ \varphi \ } & X \\[2mm]
\downarrow & & \downarrow \\[2mm]
\Sigma_g \# \Sigma_h & \longrightarrow & \Sigma_g.
\end{array}
$$

Clearly, the class $\varphi^* \Omega$ is aspherical. Furthermore, $\varphi^* \Omega$ is TNCZ since φ is a fiber map, and so $\langle \varphi^* \Omega, [F] \rangle = \langle \Omega, [F] \rangle$. Now the statement follows from Lemma 3.2. $\qquad\square$

4. Fundamental groups of fiber sums

Definition 4.1. A *short surjectivity diagram* is a diagram of groups and homomorphism of the form

(4.1)
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{j} & B & \longrightarrow & G & \longrightarrow & 1 \\
 & & \varphi \downarrow & & \downarrow \psi & & \| & & \\
1 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & G & \longrightarrow & 1
\end{array}
$$

where φ is an epimorphism and the rows are exact.

Proposition 4.2. *In the short surjectivity diagram* (4.1) *the map* ψ *yields an isomorphism* $B/j(\operatorname{Ker}\varphi) \to Q$. *Furthermore, if the top exact sequence splits then so does the bottom sequence.*

Proof. The first claim is clear by diagram chasing, cf. [M, Lemma II.3.2]. The second claim follows from the first one. □

Proposition 4.3. *Let* $F \xrightarrow{i} X \xrightarrow{f} S^2$ *be a symplectic Lefschetz fibration that admits a section. Then, for every regular fiber* F, *the inclusion* $X \setminus F \to X$ *induces an isomorphism of fundamental groups.*

Proof. Let $D = D^2 \subset S^2$ be a small closed disk centered at $f(F)$, and let $U = f^{-1}(D)$. Then $\partial U = S^1 \times \pi_1(F)$. Take the section $s : S^2 \to X$ and restrict it onto $\partial(S^2 \setminus D) = \partial D$. Now, we regard the (pointed) homotopy class α of $s : \partial D \to \partial U$ as an element of $\pi_1(\partial U)$, and it is clear that $\pi_1(\partial U) \cong \mathbb{Z} \times \pi_1(F)$ where $\mathbb{Z}$ is a subgroup generated by α.

Because of the Seifert–van Kampen theorem, we have an isomorphism

$$
\pi_1(X) \cong \pi_1(X \setminus F) *_{\pi_1(\partial U)} \pi_1(U) = \pi_1(X \setminus U) *_{\mathbb{Z} \times \pi_1(U)} \pi_1(U).
$$

Since the fibration has the global section, we conclude that the subgroup $\mathbb{Z} = \{\alpha\}$ maps to zero under both inclusions $\partial U \subset X \setminus U$ and $\partial U \subset \overline{U}$. Thus

$$
\pi_1(X) = \pi_1(X \setminus F) *_{\pi_1(U)} \pi_1(U) = \pi_1(X \setminus F) *_{\pi_1(F)} \pi_1(F) = \pi_1(X \setminus F).
$$

The proof is completed. □

Lemma 4.4. *Let* $F \xrightarrow{i} X \to S^2$ *be a symplectic Lefschetz fibration that admits a section. Let*

(4.2)
$$
F \to Y \xrightarrow{p} \Sigma_e
$$

be a surface bundle with $e > 1$ that admits a section. Then there exists a short surjectivity diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_e & \longrightarrow & 1 \\
 & & \;\;\downarrow{\scriptstyle i_*} & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X \#_F Y) & \longrightarrow & \pi_e & \longrightarrow & 1
\end{array}
$$

where the upper row is the segment of the homotopy exact sequence of the bundle (4.2).

Proof. Let f be the genus of F and

$$\left\langle a_1, b_1, \ldots, a_f, b_f \,\Big|\, \prod [a_i, b_i] \right\rangle$$

be the standard presentation of π_f. Notice that i_* is surjective because of the exactness of the sequence

$$\pi_1(F) \to \pi_1(E) \to \pi_1(B)$$

for every Lefschetz fibration $F \to E \to B$, see [GS, Proposition 8.1.9 and p. 510]. So, $\pi_1(X)$ has a presentation

$$\left\langle \widehat{a}_1, \widehat{b}_1, \ldots, \widehat{a}_f, \widehat{b}_f \,\Big|\, \prod [\widehat{a}_i, \widehat{b}_i], \widehat{R} \right\rangle$$

where $\widehat{R}$ is a finite set of words and $\widehat{a}_k = i_* a_k, \widehat{b}_k = i_* b_k, k = 1, \ldots, f$.

Let $x_1, y_1, \ldots x_e, y_e$ be the standard system of meridians and parallels on $\sigma(\Sigma_e)$, where $\sigma : \Sigma_e \to Y$ is a section of the bundle (4.2). Regarding x_i and y_i as elements of $\pi_1(Y \setminus F)$, we obtain a presentation

$$\left\langle x_1, y_1, \ldots x_e, y_e, a_1', b_1', \ldots, a_f', b_f' \,\Big|\, \prod [a_i', b_i'], q_1', \ldots, q_n' \right\rangle$$

of $\pi_1(Y \setminus F)$, where $a_k' = j_* a_k, b_k' = j_* b_k, k = 1, \ldots, f$. In fact, $\pi_1(Y \setminus F)$ is the semidirect product $\pi_1(\Sigma_e \setminus \mathrm{pt}) \ltimes \pi_1(F)$.

Take a small disk $D' \subset \Sigma_e$ about a regular value of p and consider $u' = p^{-1}(D') \cong D' \times F$. The map $\sigma : \partial D' \to \partial U' = \partial D' \times F$ gives us an element $\beta \in \pi_1(\partial U')$. Notice that the image of β in $\pi_1(Y \setminus F)$ is equal to $\displaystyle\prod_{i=1}^{e} [x_i, y_i]$.

In order to perform the fiber sum $X \#_F Y$ we identify (fiberwisely) the neighborhood U' with the neighborhood U from Proposition 4.3. It turns out to be that, under this identification, β coincides with α described in Proposition 4.3. This is true because both sections $s : \partial D \to X$ and $\sigma : \partial D' \to Y$ extend to the whole discs D and D', respectively.

Consider the group $\mathbb{Z} = \{\alpha\} = \{\beta\}$. Because of the isomorphism $\pi_1(X \setminus F) \cong \pi_1(X)$ from Proposition 4.3 and by the Seifert – van Kampen theorem, the fundamental group $\pi_1(X\#_F Y)$ of the fiber sum is the amalgamated product

$$\begin{array}{ccc}
\mathbb{Z} \times \pi_f & \xrightarrow{\overline{i}} & \pi_1(Y \setminus F) \\
{\scriptstyle \overline{j}} \downarrow & & \downarrow \\
\pi_1(X) & \longrightarrow & \pi_1(X\#_F Y)
\end{array}$$

where

$$\overline{i}(\alpha, e_1) = \Pi_{i=1}^e [x_i, y_i], \quad i(0, a_i) = a'_i, \quad \overline{i}(0, b_i) = b'_i$$
$$\overline{j}(1, e_1) = e_2, \quad \overline{j}(0, a_i) = \widehat{a}_i, \quad \overline{j}(0, b_i) = \widehat{b}_i$$

and e_1, e_2 are the neutral elements of π_f and $\pi_1(X)$, respectively. Thus the presentation of $\pi_1(X\#_F Y)$ has the following form

$$\left\langle a_1, b_1, \ldots, a_f, b_f, x_1, y_1, \ldots, x_e, y_e \,\middle|\, \Pi[a_i, b_i], \Pi[x_i, y_i], q_1, \ldots, q_n, R \right\rangle$$

where q_i are monomials q'_i with a'_i and b'_i replaced by a_i and b_i, respectively, and R consists of the words of $\widehat{R}$ with $\widehat{a}_i$ and $\widehat{b}_i$ replaced by a_i and b_i, respectively. This implies the existence of the required short surjectivity diagram. □

Corollary 4.5. *Let $F \xrightarrow{i} X \to S^2$ be a symplectic Lefschetz fibration. Let $Y = \Sigma_e \times F$. Then $\pi_1(X\#_F Y) \simeq \pi_1(X) \times \pi_e$.*

Proof. Consider the diagram of Lemma 4.4. Clearly, its top exact sequence splits, and therefore the bottom exact sequence splits by Proposition 4.2. □

Corollary 4.6. *Assume that a group Γ admits a finite representation $p_\Gamma : \pi_g \to \Gamma$ such that*

$$p_\Gamma^* : H^2(\Gamma; \mathbb{R}) \to H^2(\pi_g; \mathbb{R})$$

is a non-zero homomorphism. Then the group $\Gamma \times \pi_e$ is symplectically aspherical for all $e > 0$.

Proof. Consider the symplectic Lefschetz fibration $F \to X \to S^2$ as in Proposition 3.1. Let $Y = \Sigma_e \times F$. Then the manifold $X\#_F Y$ is symplectically aspherical by Proposition 3.3, and the claim follows from Corollary 4.5 □

5. Symplectically aspherical abelian groups

In this section we give a complete description of symplectically aspherical abelian groups.

Lemma 5.1. *Let Γ be a finitely presentable abelian group such that $H^2(\Gamma; \mathbb{R}) \neq 0$. Then there exists a finite presentation $p_\Gamma : \pi_g \to \Gamma$ such that the map*

$$p_\Gamma^* : H^2(\Gamma; \mathbb{R}) \to H^2(\pi_g; \mathbb{R})$$

is non-zero.

Proof. First, it is easy to construct a finite presentation $\pi_h \to \Gamma$ (and here there is no necessity to assume Γ abelian). Indeed, there are two finite presentations $F_r \to \Gamma$ and $\pi_{2r} \to F_r$ where F_r is the free group of r generators. Now, the composite $f : \pi_h \to F_r \to \Gamma, h = 2r$ is a finite presentation. Since $H^2(\Gamma; \mathbb{R}) \neq 0$, we conclude that there exists a monomorphism $g : \mathbb{Z}^2 \to \Gamma$ such that the induced homomorphism

$$g^* : H^2(\Gamma; \mathbb{R}) \to H^2(\mathbb{Z}^2; \mathbb{R})$$

is non-trivial. Consider the map

$$\varphi : \pi_h * \mathbb{Z}^2 \to \Gamma, \quad \varphi(a_1 b_1 \cdots a_n b_n) = f(a_1)g(b_1) \cdots f(a_n)g(a_n).$$

We claim that it is a finite presentation of Γ. Notice the following:

(1) If $f : A \to B$ and $g : B \to C$ are finite presentations then so is $gf : A \to C$.
(2) If A is a finitely presentable group then the abelianization ab $: A \to A_{\mathrm{ab}}$ is a finite presentation.

Clearly, $\pi_h * \mathbb{Z}^2$ is a finitely presentable group. Now, since Γ is an abelian group, the epimorphism φ can be decomposed as

$$\varphi : \pi_h * \mathbb{Z}^2 \xrightarrow{\mathrm{ab}} (\pi_h * \mathbb{Z}^2)_{\mathrm{ab}} \longrightarrow \Gamma$$

where both maps are finite presentations (the last one because $(\pi_h * \mathbb{Z}^2)_{\mathrm{ab}}$ is a finitely generated abelian group). So, φ is a finite presentation by (1) and (2).

Now, there is a canonical map $u : \Sigma_{h+2} \to \Sigma_h \vee T^2$ (namely, we regard Σ_{h+2} as the connected sum $\Sigma_h \# T^2$ and pinch the circle which we glued the surfaces along). Clearly, $u_* : \pi_1(\Sigma_{h+2}) \to \pi_1(\Sigma_h \vee T^2)$ is a finite presentation (its kernel is generated as a normal subgroup by the pinched circle). Thus, by (1), the homomorphism

$$\pi_{h+2} = \pi_1(\Sigma_{h+2}) \xrightarrow{u_*} \pi_1(\Sigma_h) * \pi_1(T^2) =\!\!=\!\!= \pi_h * \mathbb{Z}^2 \xrightarrow{\varphi} \Gamma$$

is the desired finite presentation of Γ. $\qquad\square$

Proof of Theorem 1.2. Let T be a finite abelian group and $\Gamma = \mathbb{Z}^m \oplus T$ with $m \geq 4$. Let $A = \mathbb{Z}^{m-2} \oplus T$. Then $H^2(A; \mathbb{R}) \neq 0$, and so, by Lemma 5.1 and Corollary 4.6 applied to the case $e = 1$, we conclude that the group $\Gamma = A \oplus \mathbb{Z}^2$ is symplectically aspherical.

Furthermore, the group $\mathbb{Z}^3 \oplus T$ is not symplectically aspherical because it is three dimensional (see Example 1.3). Finally, suppose that the group $\mathbb{Z}^2 \oplus T = \pi_1(M)$ for M closed symplectically aspherical. Since $H^i(\mathbb{Z}^2 \oplus T) = 0$ for $i > 2$, we conclude that M must be 2-dimensional by [IKRT, Proposition 2.3]). So, M is a closed surface (in fact, the torus $\mathbb{T}^2$), and thus $T = 0$. □

Corollary 5.2. *If $m > 3$, then every abelian group $\mathbb{Z}^m \oplus T$ with T finite can be realized as the fundamental group of a closed symplectically aspherical manifold M^{2n} with $4 \leq 2n \leq m$ and cannot be realized as the fundamental group of a closed symplectically aspherical manifold M^{2n} with $2n > m$.*

Proof. Take n with $4 \leq 2n \leq m$. We know that there exists 4-dimensional manifold N with $\pi_1(N) = \mathbb{Z}^{m-2n+4} \oplus T$. Now, let $M = N \times \mathbb{T}^{2n-4}$.

The last claim follows from [IKRT, Proposition 2.3]. □

Proposition 5.3. *Let N be a closed 4-dimensional symplectically aspherical manifold such that $\pi_1(N) = \mathbb{Z}^4 \oplus \mathbb{Z}/2$. Then $\pi_2(N) \neq 0$.*

Notice that Theorem 1.2 guarantees the existence of such manifold N.

Proof. Indeed, in case $\pi_2(N) = 0$ we have the Hopf exact sequence

$$\pi_3(N) \longrightarrow H_3(N) \longrightarrow H_3(\pi_1(N)) \longrightarrow 0.$$

But $H_3(N) = H^1(N) = \operatorname{Hom}(\mathbb{Z}^4 \oplus \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}^4$ by the Poincaré duality and the universal coefficient theorem, while

$$H_3(\mathbb{Z}^4 \oplus \mathbb{Z}/2) \supset H_3(\mathbb{Z}^4) \oplus H_3(\mathbb{Z}/2) = \mathbb{Z}^4 \oplus \mathbb{Z}/2.$$

Hence, there are no epimorphisms $H_3(N) \to H_3(\pi_1(N))$, and thus $\pi_2(N) \neq 0$. □

Question 8.3.2 in [IKRT] asks whether there exists a closed symplectically aspherical manifold M with $\pi_1(M) = \mathbb{Z}^4$ and $\pi_2(M) \neq 0$. Now we can answer affirmatively.

Corollary 5.4. *There exists a closed 4-dimensional symplectically aspherical manifold with $\pi_1(M) = \mathbb{Z}^4$ and $\pi_2(M) \neq 0$.*

Proof. Let N be a manifold considered in Proposition 5.3 and ω be a symplectically aspherical form on N. Let $p : M \to N$ be a two-sheeted covering with $\pi_1(M) = \mathbb{Z}^4$. Then $p^*\omega$ is a symplectically aspherical form on a closed manifold M, while $\pi_2(M) \neq 0$ and $\pi_1(M) = \mathbb{Z}^4$. $\square$

Question 8.3.1 in [IKRT] asks whether every symplectically aspherical group can be realized as the fundamental group of a closed symplectically aspherical manifold M with $\pi_2(M) = 0$.

Corollary 5.5. *If N is a closed symplectically aspherical manifold such that $\pi_1(N) = \mathbb{Z}^4 \oplus \mathbb{Z}/2$, then $\pi_2(N) \neq 0$.*

Proof. Indeed, because of Corollary 5.2, we must have $\dim N = 4$. But then $\pi_2(N) \neq 0$ by Proposition 5.3. $\square$

References

[ABCKT] J. Amorós, M. Burger, K. Corlette, D. Kotschik, D. Toledo *Fundamental groups of compact Kähler manifolds.* American Mathematical Society, Providence, RI 1996.

[ABKP] J. Amorós, F. Bogomolov, L. Katzarkov, T. Pantev (with an appendix by Ivan Smith), Symplectic Lefschetz fibrations with arbitrary fundamental groups, *J. Differential Geom.* **54**, 3 (2000), 489–545.

[ADK] D. Auroux, S. Donaldson, L. Katzarkov. Singular Lefschetz pencils. *Geom. Topol.* **9** (2005), 1043–1114.

[D1] S. Donaldson, *Lefschetz fibrations in symplectic geometry.* Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 309–314

[D2] S. Donaldson, Symplectic submanifolds and almost-complex geometry, *J. Diff. Geom.* **44** (1996) 666–705.

[F] A. Floer, Symplectic fixed points and holomorphic spheres, *Comm. Math. Phys.* **120** (1989) 575–611.

[G1] R. Gompf, A new construction of symplectic manifolds, *Annals of Math.* **142** (1995) 527–595.

[G2] R. Gompf, On symplectically aspherical symplectic manifolds with nontrivial π_2, *Math. Res. Letters* **5** (1999) 599–603.

[GS] R. Gompf, A. Stipsicz, 4-*manifolds and Kirby calculus*, Graduate Studies in Mathematics **20**, American Mathematical Society, Providence, RI (1999).

[IKRT] R. Ibáñez, J. Kȩdra, Yu. Rudyak, A. Tralle On fundamental groups of symplectically aspherical manifolds, *Math. Z.* **248**, 4, (2004) 805–826.

[M] S. Mac Lane *Homology*, Springer, Berlin, 1995.

[S] M. Schwarz, On the action spectrum for closed symplectically aspherical manifolds, *Pacif. J. Math.* **193** (2000) 419-461.

[Se] J.-P. Serre, *Trees*, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation, Springer, Berlin, 2003;

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